

Minimal Convex Decompositions

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July 17, 2012

Abstract

Let P be a set of n points on the plane in general position. We say that a set Γ of convex polygons with vertices in P is a convex decomposition of P if: Union of all elements in Γ is the convex hull of P , every element in Γ is empty, and for any two different elements of Γ their interiors are disjoint. A minimal convex decomposition of P is a convex decomposition Γ' such that for any two adjacent elements in Γ' its union is a non convex polygon. It is known that P always has a minimal convex decomposition with at most $\frac{3n}{2}$ elements. Here we prove that P always has a minimal convex decomposition with at most $\frac{10n}{7}$ elements.

1 Introduction

Let P_n denote a set of n points on the plane in general position. We denote as $Conv(P_n)$ the convex hull of P_n and c the number of its vertices, and given a polygon α we denote as α° its interior. We say that a set $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of k convex polygons with vertices in P_n is a *convex decomposition* of P_n if:

- (C1) Every $\gamma_i \in \Gamma$ is empty, that is, $P_n \cap \gamma_i^\circ = \emptyset$ for $i = 1, 2, \dots, k$.
- (C2) For every two different $\gamma_i, \gamma_j \in \Gamma$, $\gamma_i^\circ \cap \gamma_j^\circ = \emptyset$.
- (C3) $\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_k = Conv(P_n)$.

In [5] they conjectured that for every P_n there is a convex decomposition with at most $n + 1$ elements. This was disproved in [1] giving an n -point set such that every convex decomposition has at least $n + 2$ elements. Later in this direction, in [3] they give a point set P_n on which every convex decomposition has at least $\frac{11n}{10}$ elements.

We are interested in convex decompositions of P_n with as few elements as possible. A *triangulation* of P_n is a convex decomposition $T = \{t_1, t_2, \dots, t_k\}$ on which every t_i is a triangle. In [2] they prove that any triangulation T of P_n , has a set F of at least $\frac{2n}{6}$ edges that, by removing them we obtain $|F|$ convex quadrilaterals with disjoint interiors. So $\Gamma = T \setminus F$ is a convex decomposition yielding the bound $|\Gamma| \leq \frac{11n}{6} - c - 2$. We have the following definition.

Definition 1 Let Γ be a convex decomposition of P_n . If the union of any two different elements in Γ is a nonconvex polygon, then Γ will be called minimal convex decomposition.

In [4] they show that any given set P_n always has a minimal convex decomposition with at most $\frac{3n}{2} - c$ elements. Here we improve this bound giving a minimal convex decomposition of P_n with at most $\frac{10n}{7} - c$ elements.

2 Minimal Convex Decompositions

Let $p_1 = (x_1, y_1)$ be the element in P_n with the lowest y -coordinate. If there are two points with same y -coordinate we take p_1 as the element with the smallest x -coordinate.

We label every $p \in P_n \setminus \{p_1\}$ according to the angle θ between the line $y = y_1$ and the line $\overline{p_1 p}$. The point p will be labeled p_{i+1} if it has the i -th smallest angle θ , see Figure 1(a). For $i = 3, 4, \dots, n-1$, we say p_i is negative, labeled $-$, if $p_i \in \text{Conv}(\{p_1, p_{i-1}, p_{i+1}\})^o$. Otherwise we say p_i is positive, labeled $+$. See Figure 1(b).

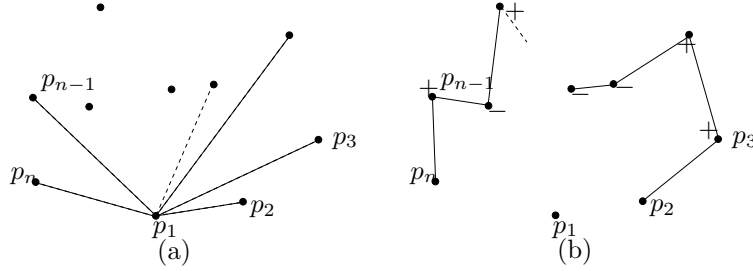


Figure 1: Labeling elements of P_n .

Let A and B be the subsets of P_n containing all positive and negative elements respectively. We divide A into subsets of consecutive points as follows:

If $p_3 \in A$, we define $A_1 = \{p_3, \dots, p_{3+r-1}\}$, as the subset with r consecutive positive points, where p_{r+3} is negative or $p_{r+3} = p_n$. If $p_3 \notin A$ then $A_1 = \emptyset$.

Suppose that $p_{n-1} \in A$. For $i \geq 2$ let $A'_i = A \setminus (A_1 \cup \dots \cup A_{i-1})$, and let $A_i = \{p_j, p_{j+1}, \dots, p_{j+r-1}\}$, where $r \geq 1$, p_j has the smallest index in A'_i , and p_{j+r} is negative or $p_{j+r} = p_n$. Let k be the number of such A_i sets obtained.

If $p_{n-1} \notin A$, we make A_{k-1} the block containing the element in A with the highest label, and then $A_k = \emptyset$.

In an analogous way we partition B into B_1, B_2, \dots, B_{k-1} . Let V be the polygon with vertex set $A \cup \{p_1, p_2, p_n\}$, and let U' be the set of at most $c-2$ regions $\text{Conv}(P_n) \setminus V$. We call U the vertex set of U' . We obtain a minimal convex decomposition Γ of P_n induced by polygons in V and U in the following way:

(1) If $A_j = \{p_i, p_{i+1}, \dots, p_{i+r-1}\}$, we make $\mathcal{A}_j = A_j \cup \{p_1, p_{i-1}, p_{i+r}\}$. \mathcal{A}_j is the vertex set of an empty convex $(|A_j| + 3)$ -gon. In case that $A_1 = \emptyset$ (or

$A_k = \emptyset$) then $\mathcal{A}_1 = \{p_1, p_2, p_3\}$ ($\mathcal{A}_k = \{p_1, p_{n-1}, p_n\}$). There are k of such polygons.

(2) If $B_j = \{p_i, p_{i+1}, \dots, p_{i+r-1}\}$, we make $\mathcal{B}_j = B_j \cup \{p_{i-1}, p_{i+r}\}$. \mathcal{B}_j is the vertex set of an empty convex $(|B_j| + 2)$ -gon. There are $k - 1$ of them.

(3) Every $B_j = \{p_i, p_{i+1}, \dots, p_{i+r-1}\}$ induces $|B_j| - 1$ triangles $\triangle p_1 p_m p_{m+1}$, for $m = i, i + 1, \dots, i + r - 2$. There are $|B_1| - 1 + |B_2| - 1 + \dots + |B_{k-1}| - 1$ of these triangles. Let T_B be the set of them.

(4) U' can be subdivided in $|A_1| + |A_2| + \dots + |A_k| - (c - 3)$ triangles with vertices in U satisfying (C1) and (C2). Make T_U the set of such triangles.

Hence, $\Gamma = \cup_i (\mathcal{A}_i \cup \mathcal{B}_i) \cup T_U \cup T_B$ is a convex decomposition of P_n . See Figure 2. We have that $|\Gamma| = k + k - 1 + |T_B| + |T_U|$ i.e.

$$|\Gamma| = n + k - c. \quad (1)$$

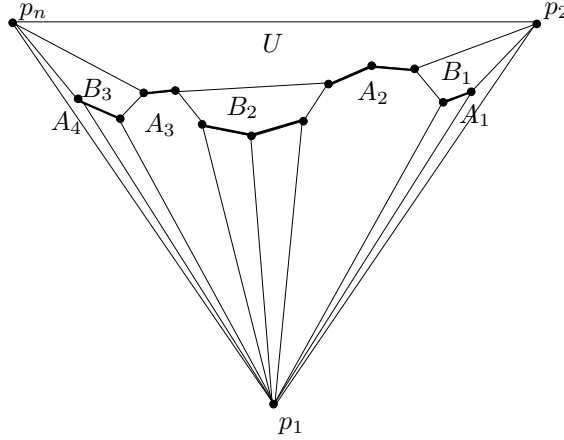


Figure 2: P_{15} and Γ described above. Here $k = 4$, so $|\Gamma| = n + k - 3 = 16$.

2.1 Convex decomposition with at most $\frac{10n}{7} - c$ elements

We proceed now to show that every collection P_n with c vertices in $\text{Conv}(P_n)$ has a convex decomposition Γ such that $|\Gamma| \leq \frac{10}{7}n - c$. We use the following notation: If in a given collection P_n we find that p_3, p_5, p_7, \dots are negative and p_4, p_6, p_8, \dots are positive, we say that P_n is a \pm set. Next result is for \pm sets.

Lemma 1 *Let P_n be a \pm set. Then P_n has a convex decomposition Γ with $\frac{4n}{3} - c$ elements, where c is the number of vertices in $\text{Conv}(P_n)$.*

Proof: For $i = 2, 8, \dots, n - 6$, we make $Q_i = \{p_1, p_i, p_{i+1}, \dots, p_{i+6}\}$, and let $T_i = \{t_1, t_2, \dots, t_9\}$ be a set of triangles with vertices in Q_i such that $t_1 = \triangle p_1 p_i p_{i+1}$, $t_2 = \triangle p_1 p_{i+1} p_{i+3}$, $t_3 = \triangle p_1 p_{i+3} p_{i+5}$, $t_4 = \triangle p_1 p_{i+5} p_{i+6}$, $t_5 = \triangle p_i p_{i+1} p_{i+2}$, $t_6 = \triangle p_{i+1} p_{i+2} p_{i+3}$, $t_7 = \triangle p_{i+2} p_{i+3} p_{i+4}$, $t_8 = \triangle p_{i+3} p_{i+4} p_{i+5}$ and $t_9 = \triangle p_{i+4} p_{i+5} p_{i+6}$, as shown in Figure 3.

We obtain a set Γ_i of convex polygons, joining elements in T_i , to get a minimal convex decomposition of P_n . We make a final modification on positive and negative points as follows: Given 3 consecutive points labeled $+$, p_i , p_j and p_k ($i < j < k$), if $p_j \in \text{Conv}(\{p_i, p_k\})^\circ$ we label p_j as $+-$, otherwise we label p_j as $++$. Analogously we modify labels $-$ to $--$ and $-+$.

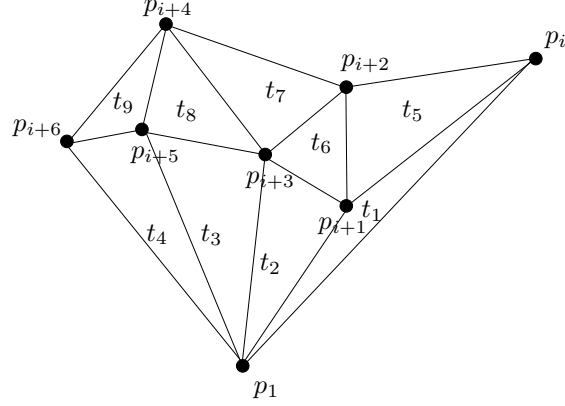


Figure 3: Q_i and its convex decomposition T_i .

We proceed now to make case analysis over the labels in p_{i+2} , p_{i+3} and p_{i+4} . Let ℓ be the line containing p_{i+1} and p_{i+3} , make \mathcal{D} the open half plane bounded by ℓ containing p_1 , and $\mathcal{U} = \mathbb{R}^2 \setminus (\mathcal{D} \cup \{\ell\})$. Given two polygons α and β sharing an edge e , we denote as $\alpha \uplus \beta$ the polygon $\alpha \cup (\beta - \{e\})$.

Case (a). p_{i+2} and p_{i+4} have label $++$ and $+-$ respectively. We have:

Subcase 1. Suppose that p_{i+3} is $--$. If $p_i \in \mathcal{D}$ and the pentagon $P = t_6 \uplus t_7 \uplus t_8$ is convex, then $\Gamma_i = \{t_1 \uplus t_2, t_3, t_4, t_5, P, t_9\}$. If P is not convex, $\Gamma_i = \{t_1 \uplus t_2, t_3 \uplus t_8, t_4, t_5, t_6 \uplus t_7, t_9\}$. See Figure 4(a).

If $p_i \in \mathcal{U}$, and the hexagon $H = t_5 \uplus t_6 \uplus t_7 \uplus t_8$ is convex, $\Gamma_i = \{t_1, t_2, t_3, t_4, H, t_9\}$. If H is not convex, $\Gamma_i = \{t_1, t_2, t_3 \uplus t_8, t_4, t_5 \uplus t_6 \uplus t_7, t_9\}$. See Figure 4(b).

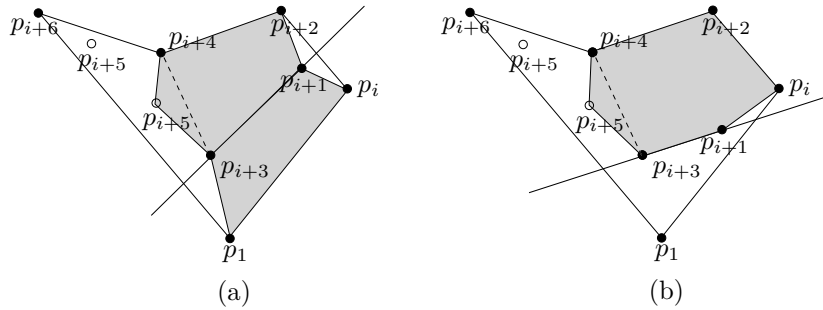


Figure 4: p_{i+2} , p_{i+4} and p_{i+3} being $++$, $+-$ and $--$ respectively.

Subcase 2. Suppose that p_{i+3} is $-+$. If p_i and p_{i+4} are in \mathcal{D} , then make $H = t_1 \uplus t_2 \uplus t_3 \uplus t_8$ and $\Gamma_i = \{H, t_4, t_5, t_6, t_7, t_9\}$ (see Figure 5(a)). Now, if $p_i \in \mathcal{U}$ (and $p_{i+4} \in \mathcal{D}$) H is missing t_1 , so $\Gamma_i = \{t_1, t_2 \uplus t_3 \uplus t_8, t_5 \uplus t_6, t_4, t_7, t_9\}$ (see Figure 5(b)). On the other hand if $p_{i+4} \in \mathcal{U}$ (and $p_i \in \mathcal{D}$), H is missing t_8 , so $\Gamma_i = \{t_1 \uplus t_2 \uplus t_3, t_4, t_5, t_6 \uplus t_7, t_8, t_9\}$ (see Figure 5(c)). Finally if $p_i, p_{i+2}, p_{i+4} \in \mathcal{U}$, $\Gamma_i = \{t_1, t_2 \uplus t_3, t_4, t_5 \uplus t_6 \uplus t_7, t_8, t_9\}$ (see Figure 5(d)).

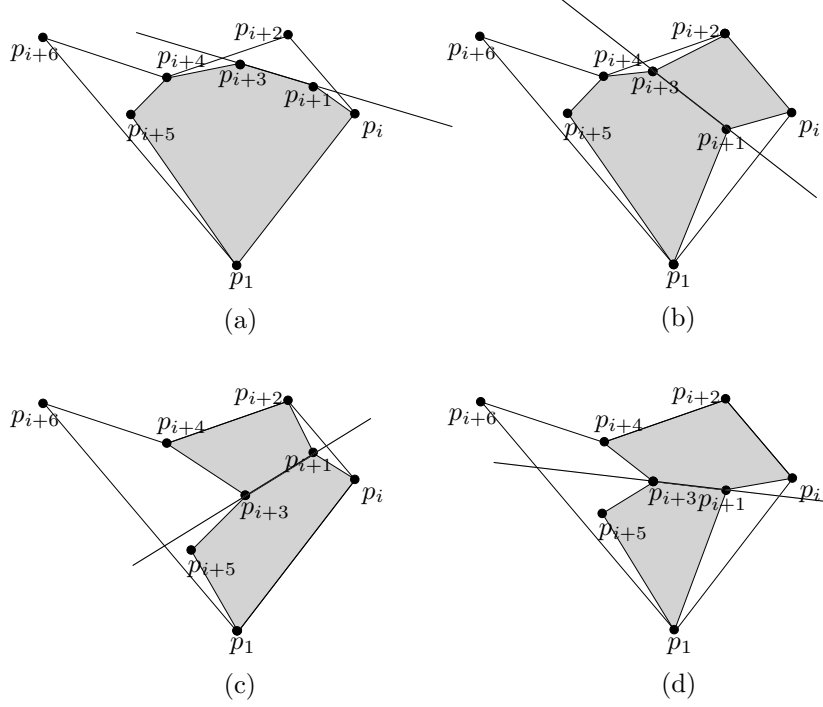


Figure 5: Polygons we find when p_{i+3} is $-+$ and p_{i+2} and p_{i+4} are $++$ and $+-$ respectively. We find same polygons if both of p_{i+2} and p_{i+4} are labeled $++$.

Case (b). Both p_{i+2} and p_{i+4} have label $+-$. Observe that $\{p_i, p_{i+2}, p_{i+4}, p_{i+6}\}$ is the set of vertices of a convex quadrilateral q , so we make $\Gamma_i = \{t_1, t_2 \uplus t_6, t_3 \uplus t_8, t_4, t_5, t_7, t_9, q\}$, $U = U \setminus \{p_{i+2}, p_{i+4}\}$ and $U' = U' \setminus q$.

Case (c). p_{i+2} and p_{i+4} both have label $++$. We are making a similar analysis as in Case (a): Suppose that p_{i+3} is $--$. If $p_i \in \mathcal{D}$ and hexagon $H = t_5 \uplus t_6 \uplus t_7 \uplus t_8$ is convex, $\Gamma_i = \{t_1, t_2, t_3, t_4, H, t_9\}$. If H is not convex, we make $\Gamma_i = \{t_1 \uplus t_2, t_3, t_4, t_5, t_6 \uplus t_7 \uplus t_8, t_9\}$. See Figure 6(a).

If $p_i \in \mathcal{U}$, H is always convex, so $\Gamma_i = \{t_1, t_2, t_3, t_4, H, t_9\}$. See Figure 6(b).

When p_{i+3} is $-+$, Γ_i has the same polygons as in subcase 2 of Case (a). And if p_{i+2} and p_{i+4} have label $+-$ and $++$ respectively, we obtain Γ_i analogously as in Case (a).

Lets make $R_i = \gamma_i \uplus \gamma_{i+1}$ where γ_i is the polygon containing t_4 in Q_i , and γ_{i+1} is the polygon containing t_1 in Q_{i+1} , and let b be the number of Q_i sets as

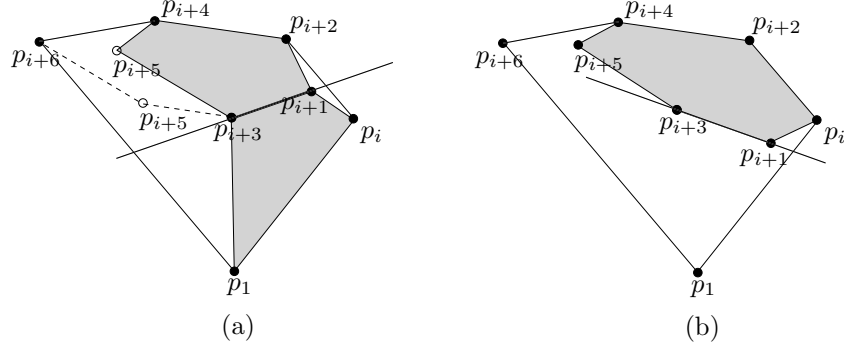


Figure 6: Polygons we find when p_{i+3} is $--$, and p_{i+2} and p_{i+4} are both $++$.

in case (b). We obtain a minimal convex decomposition of P_n by finding $\Gamma_2, \Gamma_8, \dots, \Gamma_{n-6}$, obtaining the $\frac{n}{2} - c - 2b$ triangles in T_U , and getting R_i by removing edges p_1p_i , for $i = 8, 14, 20, \dots, n - 6$.

So Γ is such that $|\Gamma| = (6\frac{n}{6} + 2b) + (\frac{n}{2} - c - 2b) - (\frac{n}{6}) = \frac{4n}{3} - c$. ■

We have the following observation.

Observation 1. Let γ be the vertex set of a convex polygon, and let p be a point in $\text{Conv}(\gamma)^o$, then $\gamma \cup p$ has a minimal convex decomposition with 3 elements.

We proceed now to prove our main theorem.

Theorem 1 Let P_n be an n -point set on the plane in general position. Then P_n has a minimal convex decomposition with at most $\frac{10}{7}n - c$ elements.

Proof: Let k be the number of polygons \mathcal{A}_i described above. If $k \leq \frac{3n}{7}$, we apply Equation (1) to find a convex decomposition with $n + k - c \leq n + \frac{3n}{7} - c$ elements. If $k = \frac{n}{2}$, P_n is a \pm set, and it has a convex decomposition with $\frac{4n}{3} - c$ elements, by Lemma 1.

In case that $\frac{3n}{7} < k < \frac{n}{2}$, we consider every \mathcal{A}_i . Let $I = \mathcal{A}_i \cap \text{Conv}(P_n)^o$. If $I = \mathcal{A}_i$ let q_i be the element in \mathcal{A}_i with the highest coordinate y (if there are 2 points with this coordinate, we make q_i the element having the greatest x coordinate of them).

If $I \neq \mathcal{A}_i$, let q_i be the element with the highest label in $\mathcal{A}_i - I$, and make r_i in B_i the element with minimum y coordinate, if there are 2 with the same coordinate, we make r_i the element with maximum x coordinate. See Figure 7.

We make $P' = \{q_1, r_1, q_2, r_2, \dots, q_{k-1}, r_{k-1}, q_k\} \cup \{p_1, p_2, p_n\}$. P' is a \pm set with $2k + 2$ elements. By Lemma 1, P' has a convex decomposition Γ' with $\frac{4}{3}(2k + 2) - c$ elements. Let S be the set $P_n - P'$, where $|S| = n - 2k - 2$. By Observation 1, we find that every element in S when is added increases in 2 the number of polygons, so P' and S induce a minimal convex decomposition Γ of $P' \cup S = P_n$ with $\frac{4}{3}(2k + 2) - c + 2|S|$ elements. Substituting $|S|$ we have that $|\Gamma| = 2n - \frac{4}{3}k - c - \frac{4}{3}$. Using the fact that $k \geq \frac{3n}{7}$ we obtain that Γ is such that $|\Gamma| \leq \frac{10n}{7} - c$. ■

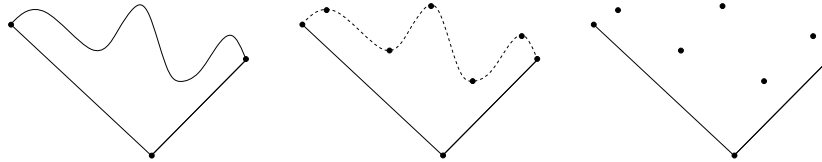


Figure 7: P_n and its collection \pm associated.

3 Concluding remarks

Analogously to triangulation of P_n , we can define *convex quadrangulation*. It would be interesting characterizing n -point sets that accept a convex quadrangulation.

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